ARTICULATING THE STUDENT MATHEMATICS IN STUDENT CONTRIBUTIONS

Laura R. Van Zoest
Western Michigan University
vanzoest@wmich.edu

Shari L. Stockero
Michigan Technological Univ
stockero@mtu.edu

Keith R. Leatham
Brigham Young University
kleatham@mathed.byu.edu

Blake E. Peterson
Brigham Young University
peterson@mathed.byu.edu

Joshua M. Ruk
Western Michigan University
jzf5431@wmich.edu

We draw on our experiences researching teachers' use of student thinking to theoretically unpack the work of attending to student contributions in order to articulate the student mathematics (SM) of those contribution. We propose four articulation-related categories of student contributions that occur in mathematics classrooms and require different teacher actions:(a) Stand Alone, which requires no inference to determine the SM; (b) Inference-Needed, which requires inferring from the context to determine the SM; (c) Clarification-Needed, which requires student clarification to determine the SM; and (d) Non-Mathematical, which has no SM. Experience articulating the SM of student contributions has the potential to increase teachers' abilities to notice and productively use student mathematical thinking during instruction.

Keywords: Classroom Discourse, Communication, Instructional Activities and Practices

Productive use of student mathematical thinking during instruction is a critical aspect of effective teaching (National Council of Teachers of Mathematics, 2014). Along with other researchers (e.g., Sherin et al., 2011), we see noticing, in particular *attending* to student mathematical thinking, *interpreting* it, and *deciding* what to do with it (Jacobs et al., 2010), as critical skills that support this productive use. Although teachers who are adept at productively using student mathematical thinking might have developed intuition and skills that allow them to notice important aspects of student contributions, such practice needs to be unpacked to support more novice teachers' learning (Boerst et al., 2011). Toward that end, in this paper we draw on our experiences researching teachers' use of student thinking to theoretically unpack the work of attending to student contributions to articulate the student mathematics in those contributions. Our goal is to contribute to the knowledge base for developing teachers' abilities to notice student mathematical thinking during instruction, abilities that lay the groundwork for productive use of that thinking.

We conceptualized a set of high-leverage instances of student mathematical thinking—what we called **M**athematically significant pedagogical **O**pportunities to build on **S**tudent **T**hinking, or MOSTs (Leatham et al., 2015). We proposed that *building* on MOSTs—turning the MOST over to the class for them to collectively make sense of it—was a productive way to use student mathematical thinking (see Van Zoest et al., 2016 for an elaboration of *building*). The first (of six) criteria for determining a MOST is "Student Mathematics" and requires evaluating a given student contribution to determine whether students' words and actions provide "sufficient evidence to make reasonable inferences" (p. 92) about what the student is saying mathematically. This way of conceptualizing *student mathematics* (SM) is how we view what it means to attend to the mathematics in student contributions. Attending to students' mathematics in this way requires attending to what students are and are not saying and being careful about the inferences we make in that regard. Such attention positions a teacher to "confidently articulate" the SM of the student contribution so that they can interpret that thinking and decide what to do with it based on the mathematics the contribution makes available for the class to engage with.

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

There is evidence that experience articulating SM can positively impact teachers' noticing. Teuscher et al. (2017) studied two pairs of student teachers, of which one pair had had experience articulating the SM for student contributions in a data set of secondary mathematics lessons. The differences between the two pairs' written reflections on student mathematical thinking during lesson observations were striking. All four consistently attended to student mathematics, but not to the same level of detail. The two who had previous experience articulating SM demonstrated skill in doing so in their reflections. The other two were only moderately able to provide a detailed articulation. This study suggests that developing teachers' skills in articulating SM positions them to attend to the mathematics of student contributions.

The Work of Articulating: Four Categories of Student Contributions

According to our view, articulating SM requires that one provide a reasoned argument for any inferences. Based on our experience justifying such inferences, we propose four distinct articulation-related categories of student contributions that require different teacher actions:

1. Stand Alone Contributions require no inference to determine the SM.
2. Inference-Needed Contributions require inferring from the context to determine the SM.
3. Clarification-Needed Contributions require student clarification to determine the SM.
4. Non-Mathematical Contributions have no SM because they are not mathematical.

Stand-Alone Contributions

Stand-alone student contributions are the easiest instances to identify the SM for, because the student contribution itself is the SM—no inference is required. This category of student contribution is straightforward and makes clear what mathematics the contribution makes available for the class to engage with. For example, consider a student contribution during an introductory lesson on adding fractions, where a student asks: "Is $\frac{2}{5}+\frac{1}{5}=\frac{3}{10}$?" The SM of this contribution is simply: *Is* $\frac{2}{5}+\frac{1}{5}=\frac{3}{10}$*?* The statement is clear (though not mathematically correct) and complete. This SM demonstrates that SM need not be true, and also that SM can come in a variety of forms, including questions.

Inference-Needed Contributions

Conversational norms dictate that we do not always use complete sentences or make explicit references. Instead, we use pronouns and take other communication shortcuts. Students do the same. Because it is impossible to know exactly what students are thinking, teachers make inferences about their students' contributions. These inferences are based on observations of what students say, gesture, and write. Thus, these shortcuts often need to be filled in to make sense of what mathematics the student contribution makes available. In these situations, although the work of inferring the SM can be done, it requires making inferences from the context. Teachers must take care, however, to not infer beyond the evidence provided by the student contribution. In particular, there is a tendency to fill in the gaps with what one wants to hear. When a student makes a contribution, it is their mathematics that needs to be attended to.

Suppose, for example, a class is asked a general question such as, "Do you understand?" and a student responds, "No." We know that the student does not understand something, but their contribution does not provide evidence of what they do not understand. In contrast, if the class was asked, "Is Ax + By = C a linear equation?," it could be reasonably inferred that if a student says "No," they actually mean *No, Ax + By = C is not a linear equation*. Here, it is reasonable to infer that the student is answering the teacher's question and the italicized statement articulates the mathematics that the contribution makes available and thus is the SM of the contribution.

In drawing such inferences, one must stay as close to the context as possible. For example, if a student says, "Can it ever have two y-intercepts?" in the context of an introductory discussion about the slope-intercept form of linear equations, a reasonably inferred SM is: *Can a graph of a linear* *equation ever have two y-intercepts?* Although it is possible that this student is wondering about the multiplicity of y-intercepts for graphs of all types of equations, the contextual evidence suggests it is more likely that they are thinking only about linear equations.

This section illustrates how articulating the SM makes explicit things that were implicit because of communication norms and the context, but does so without altering the mathematical content of the student contribution. The resulting SM is a clear articulation of a reasoned inference of what the student is expressing mathematically in the contribution.

Clarification-Needed Contributions

Clarification-needed contributions require additional information from the student to determine the SM. These contributions do not contain enough information to reasonably infer the SM; thus, we cannot reasonably articulate their SM. Sometimes clarification-needed contributions are students' attempts to articulate ideas that are particularly insightful and relevant. This is why it is critical for teachers to learn to recognize when clarification is needed and how to productively seek that clarification.

There are several ways in which a student contribution that appears mathematical may not contain enough information to reasonably infer the contribution's mathematics. For example, students often express general confusion by saying things such as, "I don't get it." Without further information we cannot reasonably infer the mathematics underlying their confusion. Sometimes students' contributions are too convoluted to make sense of what they are saying without clarification. For example, during a discussion about why ¼ times 3 is ¾, a student may state, "The 3 is like 3 and then you have a ¼." The student recognizes that there is a 3 and a ¼ involved, but how they see the relationship between these numbers is unclear. Thus, there seems to be mathematical thinking going on, but we cannot infer what it is.

Another subset of clarification-needed contributions are clarifiably ambiguous (Peterson et al., 2019). These contributions have two or more viable interpretations, and we cannot make a reasoned argument for which one best articulates the SM of the contribution. Consider the interchange when a teacher says, "Could we use unit rate to solve the proportion $\frac{6}{4} = \frac{x}{10}$?" and a student responds, "Yes, by dividing." We can infer that the student is saying, "Yes, we can use unit rate to solve the proportion $\frac{6}{4} = \frac{x}{10}$ by dividing." The latter part of the sentence, however, is ambiguous; there is no indication of which quantity would be divided by which other quantity. There are several legitimate possibilities for these quantities, resulting in multiple interpretations for this student's statement. The student might be saying "divide 6 by 4" to get 1.5 or they might be saying "divide the numerator by 2 and the denominator by 2" to simplify 6/4 to 3/2. Both of these are viable interpretations for what the student might mean by dividing. Of course, there are other possible interpretations that might reveal misconceptions about the "unit rate" strategy or about proportions in general. Thus we cannot with any level of confidence infer the SM. In order to articulate the SM of this contribution, we would need to ask the student to clarify what is being divided by what. Regardless of the reason that clarification is needed, moving forward without clarifying such contributions could lead to misunderstandings. Students could think that different ideas are being considered, leading to cross-talk and general confusion. Also, without knowing the SM of a contribution, teachers would not be able to determine whether that thinking is worth pursuing.

Non-Mathematical Contributions

Sometimes students say things like, "I need a pencil" that clearly have no mathematical content. Other times we have evidence THAT students are thinking, but there is not enough evidence to infer whether WHAT they are thinking is mathematical. For example, instances of general agreement (e.g., "Okay" or "Yeah") in response to a vague teacher question (e.g., "Does this make sense?" or

“Was this problem the same as the ones last week?”). Even when a student is engaged in mathematics, they can make contributions that have no mathematical content. For example, a student describing their graph might say, “I made my line pink because pink is my favorite color.” Non-mathematical contributions have no SM to infer.

Summary and Conclusion

We identified four categories of student contributions based on the inferability of their SM. The Stand Alone category requires no inference by a teacher because the student contribution and its SM are the same. The Inference-Needed category requires drawing on the context to infer the SM of the contribution. For both of these categories, we are able to articulate the SM of the contribution. For the Clarification-Needed and Non-Mathematical categories, we are not able to articulate an SM for the contribution; the Non-Mathematical because there is no mathematics involved and the Clarification-Needed because it needs clarification to articulate the SM. To make a Clarification-Needed contribution the focus of a whole-class discussion in its current state would likely be unproductive—at best wasting valuable instructional time and at worst introducing misconceptions. In our own work articulating the SM of student contributions from a variety of classrooms where students are given the opportunity to share their thinking, we have found many Inference-Needed and Clarification-Needed contributions in every classroom—the types of contributions that require drawing (or deciding not to draw) inferences. Thus, reflection on or observation of almost any mathematics lessons provides ample opportunities to practice this critical work of attending “within” (Stockero et al., 2017) student contributions. Attending to student contributions with the necessary precision to articulate the SM is one important aspect of the “close listening” (Confrey, 1993, p. 311) teachers need to facilitate meaningful classroom mathematics discourse. Such listening is “not mastered instantaneously” but is truly a “habit of listening” (p. 312). Experience articulating the SM of student contributions has the potential develop this habit and increase teachers’ abilities to notice student thinking during instruction.

A classroom where a wide range of student contributions are available creates a complicated environment in which to carry out the work of teaching. Teachers must continually decide which students’ ideas to make the object of a class discussion and which to respond to in other ways. In this paper we unpacked the process of figuring out what students are expressing mathematically, a foundational skill for the productive use of student thinking. Being deliberate about making reasoned inferences of the SM of a student contribution sets teachers up to make informed decisions about whether and how they use the student thinking that is available to them.

Acknowledgments

This research report is based on work supported by the U.S. National Science Foundation (NSF) under Grant Nos. DRL-1720410, DRL-1720566, and DRL-1720613. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

References

Boerst, T. A., Sleep, L., Ball, D. L., & Bass, H. (2011). Preparing teachers to lead mathematics discussions. *Teachers College Record*, 113, 2844-2877.

Confrey, J. (1993). Learning to see children's mathematics: Crucial challenges in constructivist reform. In K. Tobin (Ed.), *The practice of constructivism in science education* (pp. 299-321). Lawrence Erlbaum.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children’s mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169–202.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (2006). *Connected mathematics 2, grade 8: Variables and patterns*. Prentice Hall.

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education, 46*(1), 88–124. https://doi.org/10.5951/jresematheduc.46.1.0088

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: Author.

Peterson, B. E., Leatham, K. R., Merrill, L. M., Van Zoest, L. R., & Stockero, S. L. (2019). Clarifiable ambiguity in classroom mathematics discourse. *Investigations in Mathematics Learning*. Advance online publication. https://doi.org/10.1080/19477503.2019.1619148

Teuscher, D., Leatham, K. R., Peterson, B. E. (2017). From a framework to a lens: Learning to notice student mathematical thinking. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 31-48). Springer International. https://doi.org/10.1007/978-3-319-46753-5_3

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (2011). Situating the study of teacher noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp, R. A. (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 1–13). Routledge. https://doi.org/10.4324/9780203832714

Stockero, S. L., Leatham, K. R., Van Zoest, L. R., & Peterson, B. E. (2017). Noticing distinctions among and within instances of student mathematical thinking. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 467-480). Springer International. https://doi.org/10.1007/978-3-319-46753-5_27

Van Zoest, L. R., Peterson, B. E., Leatham, K. R., Stockero, S. L. (2016). Conceptualizing the teaching practice of building on student mathematical thinking. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1281-1288). University of Arizona.