IMPRECISION IN CLASSROOM MATHEMATICS DISCOURSE

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We theorize about ambiguity in mathematical communication and define a certain subset of ambiguous language usage as imprecise. For us, imprecision in classroom mathematics discourse hinders in-the-moment communication because the instance of imprecision is likely to create inconsistent interpretations of the same statement among individuals. We argue for the importance of attending to such imprecision as a critical aspect of attending to precision. We illustrate various types of imprecision that occur in mathematics classrooms and the ramifications of not addressing this imprecision. Based on our conceptualization of these types and ramifications, we discuss implications for research on classroom mathematics discourse.

Keywords: Classroom Discourse, Instructional Activities and Practices, Standards

The border between effective and ineffective communication of a mathematical idea can be crossed based on the use or misuse of a single word. Communication is necessarily problematic, requiring constant negotiation of meaning (Sfard & Kieran, 2001; Voigt, 1994). We never know exactly what someone else means by what they say; we infer those meanings. Communication in general, and classroom communication in particular, works because of our overall assumptions of shared meaning (Cobb, Yackel, & Wood, 1992)— for most of the words we use we assume that the individuals around us have constructed meanings that are fairly comparable with our own meanings. Though unwarranted inferences cause miscommunication, classroom discourse would come to a standstill if teachers followed every student statement with, “Please explain what you mean.” Whether consciously or not, we are constantly judging whether the words others use can stand on their own or seem to require clarification.

Those who look closely at the complexities of communicating in mathematics classrooms see one particular aspect of communication—sometimes referred to as ambiguity (e.g., Barnett-Clarke & Ramirez, 2004; Barwell, 2003)—as both inherent (and thus unavoidable) and as providing opportunities for learning. As Barwell (2003) stated, “It is the potential for ambiguity inherent in all language that allows students to investigate what it is possible to do with mathematical language, and so to explore mathematics itself” (p. 5). There is a subset of ambiguous situations, however, that we see as a barrier to mathematical communication, as hindering the negotiation of mathematical meaning in the moment. We have come to refer to such situations as imprecise. We see attention to imprecision as a critical, but possibly overlooked, aspect of the mathematical practice of attending to precision (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The practice of attending to precision often focuses on mathematical precision, like when a student uses equal in place of congruent, or when a student incorrectly strings together expressions using equal signs. Although these situations call for improved mathematical precision, we can typically infer what the student means—there is not a need to improve mathematical precision for communication to continue. We, instead, focus on language precision in the context of mathematics—when clarifying what has been said will increase the likelihood that all members of the classroom community can successfully and reliably make sense of the mathematics at hand. In this paper we define this *imprecision*, then describe and provide examples of different types of

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Pyschology of Mathematics Education*. Tucson, AZ: The University of Arizona.

imprecision. We conclude by theorizing about issues related to not addressing imprecision and ways to productively address it. We see this work as critical to our ongoing research to understand the teaching practice of productive use of student mathematical thinking.

Defining Imprecision

We now discuss an example derived from an excerpt of classroom mathematics discourse in order to motivate our definition of imprecision. In a class where students have been studying data about a group of bikers on a multi-day trip, they are examining a graph where distance is measured by the distance from a given city (see Figure 1). In a discussion about Figure 1, the class has interpreted the plotted points at times 1.5 and 2 as an indication that the bikers are stopped on the interval between 1.5 and 2 hours. A student then volunteers, “And then they went up.” The teacher asks, “What do you mean, ‘They went up?’” to which a number of students respond by making hand gestures, raising their hands up as they move from left to right.

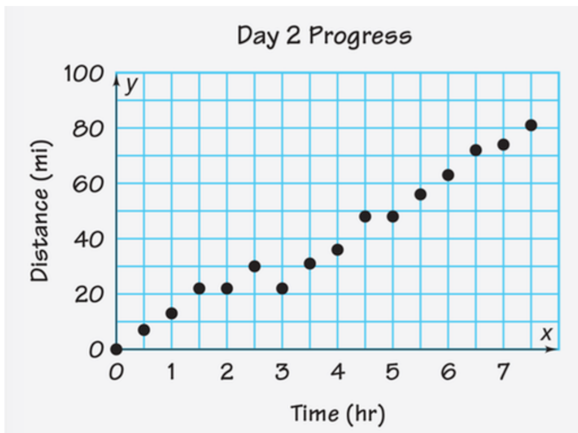

Figure 1. Bikers’ progress (from Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006, p. 12).

We see the student statement, “And then they went up,” as imprecise because it is unclear to what the student is referring by *they*. In this context *they* could easily refer to either the bikers on the road or the dots on the graph, and these two interpretations mean very different things in this context. The teacher seems to recognize a communication problem and pushes for clarification. Notice, however, that the students’ responding gestures seem to indicate their understanding of the word *up,* but the ambiguity remains. What might be missing from the teacher’s effort to clarify? We believe the ambiguity persists because the students do not know which part of the phrase *they went up* the teacher is referring to—*they* or *up*. This excerpt illustrates how imprecise language can hinder mathematical communication, creating a situation where individuals are talking past each other because of how they have interpreted the word *they*. We believe the teacher’s response could have been more productive had the request for clarification zeroed in on the specific part of the student sentence that was unclear. Had the teacher responded to the student statement with a more-specific question such as, “When you say, ‘they went up,’ what do you mean by *they*?” students would have been better positioned to clarify their meaning and the discussion could have proceeded accordingly.

Ambiguity as illustrated here creates a moment when clarification is needed in order to continue the sense-making activity, and when lack of clarification will likely cause a breakdown in the negotiation of meaning. We have come to view this type of ambiguity in the following way: If a part of speech is used such that it can be interpreted in multiple viable ways, and the existence of those interpretations causes the overall meaning of the statement in which it occurs to be unclear, then we refer to both the part of speech and the statement in which it occurs as *imprecise*. We use viable interpretations to mean interpretations some students in the class might reasonably infer given the context in which the part of speech is used. We do not consider viable extreme or outlying interpretations that are unlikely to exist in the given classroom context. Thus, for us, imprecision in classroom mathematics discourse hinders in-the-moment communication because the instance of imprecision is likely to create inconsistent interpretations of the same statement among individuals.

Types of Imprecision

In this paper, we present a theory that both researchers and practitioners can use to analyze mathematics classroom discourse. Although this is a theoretical paper, it was both prompted and informed by empirical work conducted as part of the National Science Foundation (NSF)-funded *Leveraging MOSTs* project. A central component of that project was the conceptualization of instances of student thinking teachers should take advantage of in the moment they occur during classroom discourse, what we called MOSTs—Mathematically significant pedagogical Opportunities to build on Student Thinking (Leatham, Peterson, Stockero, & Van Zoest, 2015). The associated MOST Analytic Framework provides characteristics and criteria for identifying MOSTs and for articulating why some instances fall short of being prime in-the-moment opportunities to build on student mathematical thinking. The foundational characteristic of a MOST is student mathematical thinking, and the first criterion of this characteristic is that *student mathematics* must be inferable. Thus, as we analyze student contributions to classroom mathematics discourse, we first attempt to articulate the *student mathematics* of each student's turn, or in other words, restate their comment in complete sentences or thoughts and replace pronouns and gestures with their referents when possible. In applying this framework, we often found ourselves in situations where we could not infer the student mathematics of an instance. Although at times this inability to infer the student mathematics stemmed from incomplete or inaudible statements, there were many times when student statements could be heard and seemed to be complete, but wherein we still could not make an inference. It was in this context that we came to realize that our inability to infer the student mathematics was due to language ambiguity—there were multiple viable interpretations of what the student had said. We began to wonder about these ambiguities and their effects on classroom discourse. This paper is a result of the theoretical work that followed. The examples in this paper are based on examples of imprecision we observed in middle school mathematics lessons that were analyzed for the *Leveraging MOSTs* project.

Because of the centrality of "parts of speech" to our definition, we categorize types of imprecision according to these parts of speech. The four basic parts of speech that occurred most frequently in our instances of imprecision are subject, object, adjective, and verb, and are thus our four main types of imprecision. That "parts of speech" ended up having the explanatory power that it did as we theorized about classroom imprecision was a surprisingly straightforward way of capturing what was a complex problem for us, one with which we grappled for a long time. The related grammatical phenomenon of unclear referents is a typical topic in textbooks on grammar, but not so common in research related to oral communication and learning to write and speak in general. When research does attend explicitly to unclear referents it tends to be research on those who are acquiring a second language (Block, 1992) or who have learning challenges such as delayed development (e.g., Eigsti, de Marchena, Schuh, & Kelley, 2011) or dementia (e.g., Almor, Kempler, MacDonald, Andersen, & Tyler, 1999). The theory presented in this paper contributes to the literature by extending a focus on language imprecision to typical classroom mathematics discourse and the complexity of orchestrating that discourse.

Subject

The subject of a sentence is the person, place, or thing that is doing or being something. Subjects are either nouns or pronouns and can be imprecise if it is not clear which person, place, or thing is being referenced. We first share an example of how the use of a pronoun as the subject of a sentence can be imprecise in a mathematics classroom and then an example of an imprecise demonstrative pronoun.

Pronoun. In the previous example – "And then they went up." – the subject of the sentence is they and they is imprecise because of the ambiguity of its referent. As mentioned before, in this context, they could be referring to the dots on the graph or to the bikers on the road. These alternate possibilities for the subject of the sentence result in very different mathematical interpretation for the overall meaning of the sentence. But, if we know whether they is referring to bikers or dots, it is easier to infer the meaning of went up and thus the meaning of the entire sentence.

Demonstrative pronoun. Demonstrative pronouns are pronouns such as *this, that, these,* and *those* that point to specific things. In a class discussion about slopes of linear equations, a student says, "That has a positive slope." The precision of this statement depends on the context in which it occurs. If there is a single linear equation on the board or being talked about, then it is likely clear what subject is being referenced and there is no imprecision. If, however, there are multiple linear equations on the board and only one of them has a positive slope, then the student statement is imprecise because the demonstrative pronoun *that* could be referring to either of those equations and thus there are multiple viable interpretations of the statement—the meaning behind the student statement is very different depending on which equation they are referencing. Were the teacher to seek clarification here, awareness that the imprecision lies with the demonstrative pronoun that is the subject of the sentence might lead to a question such as, "What has a positive slope?"

Object

Similar to the subject of a sentence, objects are either nouns or pronouns and are imprecise when it is not clear which person, place or thing is being referenced. If the object is a pronoun or demonstrative pronoun, the imprecision can occur in the same way it occurs in the subject of the sentence, and clarification is best if it hones in on the imprecise object itself. In the case of an object, however, another type of imprecision can occur, one that generally does not occur with subjects (in the English language): the object of a sentence can be implied. Such imprecision can occur with objects of a verb and with objects of an adjective.

Implied object of verb. Consider the teacher-student interchange when a teacher says, "What about unit rate? Could we use unit rate to solve this proportion [6/4=f/10]?" and a student responds, "Yes, by dividing." From the context we can infer that the student is saying, "We can use unit rate to solve the proportion 6/4=f/10 by dividing." The latter part of the sentence, however, is incomplete; the verb *divide* has an implied object and therefore there is no indication of which numbers are to be divided. There are several legitimate possibilities for these numbers, not to mention several others that might reveal misconceptions about the "unit rate" strategy or about proportions in general. We thus see the statement *by dividing* as an example of imprecision because of an unclear implied object of a verb. A teacher could zero in on the part of speech that created the imprecision by asking, "Which numbers would you divide?" thus acknowledging the unclear part of the statement and pushing for an articulation of the object.

Implied object of adjective. In an 8th grade algebra class, students were learning about the composition of functions and were given two equations: P=2.50V - 500 and V=600 - 500R, where profit (P) is related to the number of visitors (V) to an amusement park, and the number of visitors (V) is related to the probability of rain (R). Students were first asked to determine the profit when the probability of rain is 25% and then to find the probability of rain when the expected profit is $625 (from Lappan et al. 2006b, p. 25). After students had worked on these problems, the teacher said, "So, when they tell you a value of a variable, you substituted that variable with the value they told you. So, you guys were okay with that part. Why do you think the second part was kind of hard?" A student responded, "Because you had to do the opposite." In this statement the word *opposite* is an adjective and we know what it means. What we do not know is what object this adjective is describing—the opposite operation, the opposite order or the opposite process. Any of these objects are viable interpretations of what the student has said. Thus the statement is imprecise because the implied object of the adjective *opposite* is imprecise.

Adjective

Adjectives, words that describe nouns, are sometimes the culprit of the imprecision. For example, a teacher might ask, "Could we still use *that* strategy?" in a context where multiple strategies have been discussed. For students to answer this question they need to know which strategy is under consideration—the demonstrative adjective *that* is imprecise. Or suppose a class is discussing a graph displaying a dozen points and a student says, "Find the distance between those points." In order to pursue this line of reasoning, and for the class to follow along, the class needs to know which of *those* points the student is considering.

The adjectives in these examples are demonstrative adjectives—in essence they are pronouns being used as adjectives. Note how slight variations on these examples change the part of speech that is imprecise. Compare "could we still use *that*" with "could we still use *that* strategy". In the first case the pronoun *that* is the object of the sentence. In the second case *that* is a demonstrative adjective. The second sentence is "less imprecise" in that we at least know that the object of the sentence is a strategy, we just do not know which strategy. This distinction matters because, whereas in the first sentence one would seek broader clarification of the object of the sentence with a question such as, "Could still use what?", in the second sentence the clarification question could be much more precise—"Could still use *which* strategy?". The more one can hone in on the part of speech that is imprecise the more precise one can be in seeking clarification.

Verb

To understand a sentence, one must understand the subject's action, or the *verb*. This understanding is particularly important in mathematics because of the many carefully defined mathematical verbs like *add*, *divide*, *solve*, *invert*, and *integrate* that are used to communicate specific actions. A common type of imprecise verb use is the use of generic, colloquial action verbs such as *work*, *do*, and *make*, which do not have precise meanings in a mathematical context and thus can often be interpreted in more than one way. In a sense, these generic action verbs are used like pronouns to replace more precise mathematical verbs.

The following example illustrates how use of generic action verbs can create imprecision. It comes from the same 8th grade algebra class mentioned previously where students were using the equations $P=2.50V - 500$ and $V=600 - 500R$. Students were able to solve for P given R relatively easily, but many struggled when asked to solve for R given P. During a conversation about that struggle, a student said, "[In the first case you] just do the equation instead of doing multiple step equations." Here the student used the verb *do* in a general, colloquial way; it is not clear what she meant by "do the equation" or "doing multiple step equations," and whether each use of *do* is the same. She may have meant solve the equation(s), evaluate the equation(s), substitute something within the equation(s), or manipulate the equation(s). Because there are multiple viable options for what was meant by *do*, the student statement was imprecise. This student seemed to have something valuable to contribute to the mathematical conversation about students' struggles with this task, but the imprecision hindered her communication. To clarify this imprecision, the teacher could ask the student to clarify what she means by *do* when she says, "*do* the equation." This teacher response would hone in on the verb imprecision while simultaneously legitimizing the student statement as an important contribution to the mathematical conversation.

Commonality Across the Types

Across these various types of imprecision, one particular commonality stands out. Multiple viable interpretations occur when generic or implied words are used in place of more specific words. Pronouns are a wonderful tool for streamlining communication, but when their referents are unclear from context, imprecision occurs. Further complicating matters, the English language allows for specific subjects and objects to be completely absent, creating an even deeper layer of inference and associated possibilities for imprecision. Our examples of imprecise verbs also fit this pattern, as noted earlier, as these verbs are used in a generic way, almost like a pronoun.

Ramifications of Not Addressing Imprecision

Imprecision could cause a student to think they do not understand something when, in reality, there is just a breakdown in the negotiation of meaning because of an imprecise statement. When an imprecise statement is not explicitly addressed, students are likely left unaware that what has been said is imprecise. If the teacher implicitly infers the meaning of an imprecise statement, students are likely not aware of the sense making that the teacher has engaged in, so have no idea that their interpretation of the statement is different from the teacher or other students in the class. In addition, there seems to be a norm in classrooms that a teacher moving on implies that what was said was clear or true. If a student cannot make sense of an imprecise statement, they may not know whether they lack understanding or there was a problem with what was said. In other words, they may think their own understanding is flawed because it does not reconcile with the imprecise statement when, in fact, the imprecise statement is where the flaw lies.

Furthermore, a number of instances of imprecision we have observed have led to the creation of simultaneous, yet parallel inconsistent conversations, wherein various participants proceeded with differing interpretations of an imprecise statement, in essence talking past each other. Because imprecision occurs when a word, statement or action has more than one viable interpretation, communication is hampered when part of the class adopts one of those interpretations and another part of the class adopts a different interpretation.

These two main ramifications of imprecision—student confusion and parallel conversations—have serious implications regarding the teacher's and students' experience in the classroom. First, students might disengage from the class discourse because of their inability to make sense of an imprecise statement. When imprecision occurs and students are confused or when their understanding does not align with the teacher's, some proactive students might push on the issue until the imprecision is cleared up and the confusion is resolved. Unfortunately, this is likely the exception as many students are unwilling to challenge a teacher or stall progressing discourse. These students are likely to remain confused, ultimately causing them to disengage from the discourse because of their inability to make sense of the ensuing conversation, all caused by unaddressed imprecision. Second, these ramifications may cause the teacher to miss opportunities to better understand a student's thinking, and thus miss opportunities to further that student's and the class's understanding of the mathematics at hand. Since we began to think about imprecision as we observed it during the MOST project, we particularly emphasize this implication. When a student's utterance is imprecise and when the teacher does not address that imprecision, the teacher is not able to articulate the *student mathematics* of that student's statement with confidence. Without that understanding, they are not able to effectively further that student's thinking about the mathematics at hand, nor are they able to use that student's comment to further the rest of the class's understanding. Third, there could be repercussions for students' mathematical understanding. For instance, in the "they went up" example, failure to explicitly address the imprecision could cause or reinforce the misconception that a graph is a picture of the physical situation.

Explicitly Addressing Imprecision

Student mathematical thinking should be at the heart of classroom mathematics discourse (e.g., National Council of Teachers of Mathematics, 2014). In order for students and teachers alike to fully benefit from students making their thinking public, teachers need to recognize and then attend to roadblocks that hinder the effective communication of the intended ideas. To do this effectively, a teacher needs to attempt to internally make sense of student comments in order to recognize instances of imprecision; then it is critical to push for clarification when imprecision occurs to allow others in the classroom to also make sense of what is being said. Although it is unwise to ask for clarification about every student statement, it is equally unwise to never seek such clarification. Some teachers, particularly novice teachers, may be reluctant to push for clarification from their students because they feel such requests may come across as a lack of mathematical understanding on their part. Members of a classroom community should recognize that a push for clarification is not an indication of weak mathematical understanding, but rather an acknowledgement of the importance of clear communication and evidence of the centrality of students sharing their thinking to mathematics teaching and learning.

As we saw in the example of "they went up," even when teachers seem to recognize that something is amiss, their requests for clarification may miss the mark and thus not solve the problem. If teachers can learn to attend to precision by attuning themselves to the specific cause of imprecision (i.e., the particular part of speech that is imprecise), they can ask for clarification that hones in on just what it is the student needs to clarify. Such clarification specificity has at least three advantages. First, this specificity helps the student to know what aspect of their communication was problematic, providing guidance for them as they seek to clarify their ideas. Second, this specificity scaffolds the entire class as they try to negotiate the meaning of what has been said. Third, and perhaps most important, it sends the message that most of what a student has said has been taken as understood—as understandable and meaningful. Such messages play an important role in helping students to gain confidence in their abilities to contribute legitimate, useful mathematical thinking. By explicitly addressing an instance of imprecision, teachers legitimize all students' efforts to make sense of others' ideas; they also model the importance of attending to precision.

In conclusion, we see attending to imprecision as a critical and possibly overlooked aspect of the study of the productive use of student mathematical thinking in classroom discourse. Future research could use this conceptualization of imprecision as a tool that could help us better understand the barriers to effective classroom discourse. We believe that this tool is also readily accessible to teachers in their in-the-moment analysis of classroom discourse, thus blurring the border between research and practice.

Acknowledgements

This work was funded by the U.S. National Science Foundation (NSF) under Grant Nos. 1220141, 1220357, and 1220148. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NSF.

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